Why Math on a Saturday Afternoon?
A Medley of Algebra Problems

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1. Simplify the expression:

$$\frac{1}{x-1} - \frac{3}{x^3-1}$$

Solution. Note that the factor $x-1$ appears in both denominators, since

$$x^3 - 1 = (x-1)(x^2+x+1).$$

Hence

$$\frac{1}{x-1} - \frac{3}{x^3-1} = \frac{1}{x-1}\left(1 - \frac{3}{x^2+x+1}\right).$$

There is no mystery now: we compute

$$1 - \frac{3}{x^2+x+1} = \frac{x^2+x-2}{x^2+x+1}$$

We observe that the numerator can be factored as $(x-1)(x+2)$. Thus

$$\frac{1}{x-1} - \frac{3}{x^3-1} = \frac{1}{x-1} \cdot \frac{(x-1)(x+2)}{x^2+x+1} = \frac{x+2}{x^2+x+1}.$$

2. Evaluate

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}$$

Solution. Write $(n+1)! = (n+1) \cdot n!$. Let's substitute.

$$\frac{(6!+5!)(5!+4!)(4!+3!)(3!+2!)(2!+1!)}{(6!-5!)(5!-4!)(4!-3!)(3!-2!)(2!-1!)}$$

$$= \frac{5!(6+1) \cdot 4!(5+1) \cdot 3!(4+1) \cdot 2!(3+1) \cdot 1!(2+1)}{5!(6-1) \cdot 4!(5-1) \cdot 3!(4-1) \cdot 2!(3-1) \cdot 1!(2-1)}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6}{2 \cdot 1} = 21.$$

3. The ratio of the harmonic mean of two positive real numbers a and b, $a < b$, to their geometric mean is $12 : 13$. Find $a : b$.

Solution. We have

$$\frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{12}{13}$$

which is equivalent to

$$\frac{\sqrt{ab}}{a+b} = \frac{6}{13}.$$

It follows that

$$\frac{a+b}{\sqrt{ab}} = \frac{13}{6} \Leftrightarrow 6\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right) = 13.$$

Set $\sqrt{\frac{a}{b}} = t$. Then

$$6\left(t + \frac{1}{t}\right) = 13$$

which reduces to

$$(2t-3)(3t-2) = 0.$$

This implies $t = \frac{3}{2}$ or $t = \frac{2}{3}$. It follows that $\frac{a}{b} = \frac{4}{9}$, implying $a : b = 4 : 9$ since $a < b$.

4. Let

$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$

Compute $(x+1)^{48}$.

Solution. We start by letting $a = \sqrt[16]{5}$, so that

$$x = \frac{4}{(a^8+1)(a^4+1)(a^2+1)(a+1)}.$$

Now, we observe that the denominator appears in the factorization of $a^{16} - 1$, since

$$\begin{aligned}
&(a-1)(a+1)(a^2+1)(a^4+1)(a^8+1) \\
&= (a^2-1)(a^2+1)(a^4+1)(a^8+1) \\
&= (a^4-1)(a^4+1)(a^8+1) \\
&= (a^8-1)(a^8+1) = a^{16} - 1
\end{aligned}$$

repeatedly applying difference of squares.

Hence we conclude that

$$x = \frac{4(a-1)}{a^{16}-1} = \frac{4(a-1)}{5-1} = a - 1.$$

Thus

$$(x+1)^{48} = a^{48} = (a^{16})^3 = 5^3 = 125,$$

showing that the answer of the problem is 125.

5. Evaluate the product

$$\prod_{k=1}^{n}\left(1 + \frac{2^k}{1+2^k}\right).$$

Solution. Rewrite the product as

$$\prod_{k=1}^{n}\left(1 + \frac{2^k}{1+2^k}\right) = \prod_{k=1}^{n}\frac{1+2^{k+1}}{1+2^k}$$

This product telescopes, and is equal to $\frac{1+2^{n+1}}{3}$.

6. Solve the equation

$$\frac{36^x}{54^x - 24^x} = \frac{6}{5}$$

Solution. Set $2^x = a$ and $3^x = b$. The equation becomes

$$\frac{a^2b^2}{ab^3 - a^3b} = \frac{6}{5}$$

which is equivalent to

$$6\left(\frac{b}{a} - \frac{a}{b}\right) = 5.$$

The substitution $y = \frac{b}{a} = \left(\frac{3}{2}\right)^x$ yields $6y - \frac{6}{y} = 5$, that is $6y^2 - 5y - 6$.

The solutions to this quadratic equation are $y_1 = \frac{3}{2}, y_2 = \frac{2}{3}$. Since $y = \left(\frac{3}{2}\right)^x$, we obtain the solutions $x = 1$ or $x = -1$. Checking, we find only $x = 1$ works.

7. Prove that for any real numbers a and b,

$$ab - 3 \leq (a + b + 3)^2.$$

Solution. Set $a + b = s, ab = p$. The inequality $(a - b)^2 \geq 0$ implies $(a + b)^2 \geq 4ab$ which implies $p \leq \frac{s^2}{4}$.

Hence it suffices to show that

$$\frac{s^2}{4} - 3 \leq (s + 3)^2.$$

This inequality is equivalent to

$$s^2 - 12 \leq 4s^2 + 24s + 36$$

which reduces to $3(s + 4)^2 \geq 0$, which is true! Equality holds if and only if $s = -4$ and $a = b$, which occurs if and only if $a = b = -2$.

8. Evaluate

$$\prod_{k=1}^{n} \left(\frac{1}{8} + \frac{k+1}{(2k+1)^2}\right).$$

Solution. Within each term of the product, create a common denominator. We have

$$\prod_{k=1}^{n} \left(\frac{1}{8} + \frac{k+1}{(2k+1)^2}\right) = \prod_{k=1}^{n} \frac{1}{8}\left(\frac{(2k+3)^2}{(2k+1)^2}\right).$$

This is a telescoping product, which equals

$$\frac{1}{8^n}\left(\frac{2n+3}{3}\right)^2.$$

9. Solve in real numbers the system of equations

$$\begin{cases} x^2 + 7 = 5y - 6z \\ y^2 + 7 = 10z + 3x \\ z^2 + 7 = -x + 3y. \end{cases}$$

Solution. Summing up the three equations we obtain

$$x^2 + y^2 + z^2 + 21 = 2x + 8y + 4z$$

which can be written as

$$(x-1)^2 + (y-4)^2 + (z-2)^2 = 0.$$

Since a sum of squares can only equal zero if all of the terms are zero, $(x, y, z) = (1, 4, 2)$ is the only solution.

10. Let a be a positive real number such that

$$\frac{a^2}{a^4 - a^2 + 1} = \frac{4}{37}.$$

Compute $\dfrac{a^3}{a^6 - a^3 + 1}$.

Solution. Taking the reciprocal, write the given equation as

$$a^2 - 1 + \frac{1}{a^2} = \frac{a^4 - a^2 + 1}{a^2} = \frac{37}{4}.$$

We need to compute

$$\frac{a^3}{a^6 - a^3 + 1} = \frac{1}{a^3 - 1 + \frac{1}{a^3}}.$$

This strongly suggests considering $x = a + \frac{1}{a}$ as the real variable of the problem.
Since

$$a^2 + \frac{1}{a^2} = x^2 - 2$$

and

$$x^3 = a^3 + \frac{1}{a^3} + 3a + 3\frac{1}{a} = 3x + a^3 + \frac{1}{a^3},$$

the first equation gives

$$x^2 = 3 + \frac{37}{4} = \frac{49}{4}.$$

Since a is positive, so is x, thus the previous relation is equivalent to $x = \frac{7}{2}$.
We can therefore compute

$$\frac{a^3}{a^6 - a^3 + 1} = \frac{1}{a^3 - 1 + \frac{1}{a^3}} = \frac{1}{x^3 - 3x - 1} = \frac{1}{\frac{251}{8}} = \frac{8}{251}.$$

11. Let a, b, c be nonzero complex numbers such that

$$a - \frac{1}{b} = 3, \quad b - \frac{1}{c} = 4, \quad c - \frac{1}{a} = 5.$$

Find $abc - \frac{1}{abc}$.

Solution. We symmetrize the problem by multiplying the given relations:

$$\left(a - \frac{1}{b}\right)\left(b - \frac{1}{c}\right)\left(c - \frac{1}{a}\right) = 60.$$

Expanding brutally the left hand-side, we get the equivalent relation

$$abc - \frac{1}{abc} - a - b - c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 60.$$

On the other hand, adding the given relations yields

$$a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}=12.$$

Finally, adding the two previous equalities, we obtain

$$abc-\frac{1}{abc}=72.$$

12. Let $a\ge b\ge c>0$. Prove that

$$(a-b+c)\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\right)\ge 1.$$

Solution. We write the inequality as follows

$$\frac{1}{a}-\frac{1}{b}+\frac{1}{c}\ge\frac{1}{a-b+c}.$$

It is equivalent to

$$\frac{a+c}{ac}\ge\frac{a+c}{b(a-b+c)}.$$

Therefore it is enough to check that

$$ac\le b(a-b+c)\quad\text{or}\quad(b-a)(b-c)\le 0.$$

The last inequality is true due to the given condition. Hence, the proof is completed.

13. Let a,b,c be side lengths of a triangle. Prove that

$$\frac{a}{a-b+c}+\frac{b}{b-c+a}+\frac{c}{c-a+b}\ge 3.$$

Solution. We make the denominators nice by making a substitution that takes advantage of the symmetry.

Let $a-b+c=x, b-c+a=y, c-a+b=z$.

Adding the first two equations yields $2a=x+y$, and similarly we have $2b=y+z, 2c=z+x$.

Multiplying both sides of the inequality by 2 and substituting, the inequality reduces to

$$\frac{x+y}{x}+\frac{y+z}{y}+\frac{z+x}{z}\ge 6.$$

Canceling the constant terms, the inequality becomes

$$\frac{z}{y}+\frac{x}{z}+\frac{y}{x}\ge 3$$

which is true by AM-GM.

(AM-GM is justified because x,y,z are positive, since a,b,c are sides of a triangle!)

14. Let $f(x)=\dfrac{1}{1+3^{2x-1}}$. Evaluate

$$f\left(\frac{1}{2013}\right)+f\left(\frac{2}{2013}\right)+f\left(\frac{3}{2013}\right)+\cdots+f\left(\frac{2012}{2013}\right).$$

Solution. Observe that

$$f(1-x)=\frac{1}{1+3^{1-2x}}=\frac{3^{2x-1}}{3^{2x-1}+1}.$$

It follows that $f(x) + f(1-x) = 1$ for all x. Then, using this symmetry, pairing outer terms, we have

$$f\left(\frac{1}{2013}\right) + f\left(\frac{2012}{2013}\right) = 1, f\left(\frac{2}{2013}\right) + f\left(\frac{2011}{2013}\right) = 1, \cdots$$

It follows that the sum is equal to $\frac{2012}{2} = 1006$.

15. Let a, b, c, d be real numbers greater than 0 satisfying $abcd = 1$. Prove that

$$\frac{1}{a+b+2} + \frac{1}{b+c+2} + \frac{1}{c+d+2} + \frac{1}{d+a+2} \geq 1.$$

Solution. By AM-GM, we have $a + b \geq 2\sqrt{ab}$. Then

$$\frac{1}{a+b+2} + \frac{1}{c+d+2} \geq \frac{1}{2\sqrt{ab}+2} + \frac{1}{2\sqrt{cd}+2}.$$

Let $\sqrt{ab} = x$. By the given condition, it follows that $\sqrt{cd} = \frac{1}{x}$. Then

$$\frac{1}{2\sqrt{ab}+2} + \frac{1}{2\sqrt{cd}+2} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{x}{1+x}\right) = \frac{1}{2}.$$

Similarly,

$$\frac{1}{b+c+2} + \frac{1}{d+a+2} \geq \frac{1}{2}.$$

The conclusion follows by adding the two inequalities with equality when $a = b = c = d = 1$.